

Solutions to Tutorial 1

1 Market-implied probabilities

- (a) BioTech trades at $S_0 = 100$. At the end of its clinical trial, its share price will be either $S_T = 400$ if the drug passes or $S_T = 50$ if it fails. Assuming the price S_0 is fair, what is the market-implied probability of the test succeeding?
- (b) BioTech must now complete two trial stages. Each stage has the same market-implied pass probability q , and the stages are independent under the pricing measure. The terminal share price depends on both results:

Stage 1	Stage 2	S_T
fail	fail	50
fail	pass	80
pass	fail	120
pass	pass	400

The shares trade at $S_0 = 100$.

- (i) Draw the two-stage event tree and assign a probability to each terminal outcome.
- (ii) Determine q and compute the market-implied probability of each of the four terminal outcomes.
- (a) Let q be the market-implied probability of success. Under the exercise's undiscounted fair-price convention,

$$100 = 400q + 50(1 - q), \quad q = \frac{1}{7}.$$

- (b) Each stage passes independently with probability q .

- (i) The terminal probabilities are

$$\begin{array}{l|l} FF & (1 - q)^2 \\ FP & (1 - q)q \\ PF & q(1 - q) \\ PP & q^2 \end{array}.$$

- (ii) Fair pricing gives

$$50(1 - q)^2 + 80(1 - q)q + 120q(1 - q) + 400q^2 = 100,$$

or $5q^2 + 2q - 1 = 0$. The only root in $[0, 1]$ is

$$q = \frac{\sqrt{6} - 1}{5} \approx 0.2899.$$

Thus

$$(p_{FF}, p_{FP}, p_{PF}, p_{PP}) = ((1 - q)^2, (1 - q)q, q(1 - q), q^2) \approx (0.5042, 0.2059, 0.2059, 0.0840).$$

2 Easy derivatives

Compute the following derivatives.

(a) $\frac{d \tan(x)}{dx}.$

(b) $\frac{d \sin(x^2)}{dx}.$

(c) $\frac{d}{dx} \sqrt{\frac{x+1}{x-1}}.$

(a) $\frac{d}{dx} \tan(x) = \sec^2(x).$

(b) By the chain rule,

$$\frac{d}{dx} \sin(x^2) = 2x \cos(x^2).$$

(c) For $g(x) = (x+1)/(x-1)$, we have $g'(x) = -2/(x-1)^2$. Hence

$$\frac{d}{dx} \sqrt{\frac{x+1}{x-1}} = -\frac{1}{(x-1)^2 \sqrt{\frac{x+1}{x-1}}},$$

for $x < -1$ or $x > 1$.

3 Expected values

Compute the following integrals over $\mathbb{R}$.

(a) $\int_{\mathbb{R}} x e^{-x^2} dx.$

(b) $\int_{\mathbb{R}} \frac{2x}{x^2 + 3x + 2} dx.$

(c) $\int_{\mathbb{R}} \frac{e^x}{1 + e^{2x}} dx.$

(a) The integrand is odd and absolutely integrable, so

$$\int_{\mathbb{R}} x e^{-x^2} dx = 0.$$

(b) Since $x^2 + 3x + 2 = (x+1)(x+2)$, the integrand has poles at $x = -2$ and $x = -1$. Thus the improper integral does not converge.

(c) Set $u = e^x$. Then u runs from 0 to ∞ , and

$$\int_{\mathbb{R}} \frac{e^x}{1 + e^{2x}} dx = \int_0^\infty \frac{1}{1 + u^2} du = \frac{\pi}{2}.$$

4 Option payoff

Compute the following integrals, where $\sigma > 0$.

(a) $\int_{\mathbb{R}} x^+ \frac{3}{x(5+2x)} dx.$

(b) $\int_{\mathbb{R}} x^+ \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx.$

(a) Since x^+/x is the indicator of $(0, \infty)$ almost everywhere, the integral becomes

$$\int_0^\infty \frac{3}{5+2x} dx = \lim_{R \rightarrow \infty} \frac{3}{2} (\ln(5+2R) - \ln 5) = \infty.$$

(b) Only $x > 0$ contributes. Therefore

$$\frac{1}{\sigma \sqrt{2\pi}} \int_0^\infty x e^{-\frac{x^2}{2\sigma^2}} dx = \frac{\sigma}{\sqrt{2\pi}}.$$